

Transverse lattice

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Abstract

The transverse lattice approach to non-perturbative light-front hamiltonian QCD is described. Preliminary results on the $\pi - \rho$ system are presented, at fixed DLCQ and Tamm-Dancoff cut-offs. A renormalised, approximately Lorentz covariant light-front hamiltonian is found to leading order of the colour-dielectric expansion, compatible with a massless pion. The π light-front wavefunction is compared with experiment. Exclusive processes agree reasonably well, given the approximations, but inclusive processes, sensitive to higher Fock state structure, still exhibit large cut-off artifacts.

1 Introduction

A light-front hamiltonian has wavefunctions initialised on a null plane $x^+ = (x^0 + x^3)/\sqrt{2} = 0$. They are manifestly Lorentz-boost invariant and therefore suitable for highly relativistic problems. Indeed, many high energy QCD scattering or decay processes factorize into a non-perturbative light-front wavefunction convoluted with a perturbative scattering kernel [1]. The calculation of light-front wavefunctions in QCD is not easy, however. One has to address the usual problems of confinement and chiral symmetry breaking in an unusual theoretical context, where renormalisation issues are non-standard and unavoidably non-perturbative.

Path integral lattice gauge theory simulation has been the most comprehensive method to compute hadronic observables from first principles QCD. Some indirect information on light-front wavefunctions has also been obtained from

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this method, but far less than one would like. It is therefore natural to attempt a light-front hamiltonian quantisation of lattice gauge theory. Treating x^+ as canonical time and $\{x^- = (x^0 - x^3)/\sqrt{2}, x^1, x^2\}$ as the spatial variables, x^+ must be continuous for a light-cone hamiltonian formulation. A high-energy lattice cut-off can only reasonably be applied to the transverse spatial directions $\mathbf{x} = \{x^1, x^2\}$, since large values of the momentum p^+ conjugate to x^- correspond to small energies p^- conjugate to x^+ . A high energy cut-off may be applied to x^- by making it periodic, $x^- \equiv x^- + L$. For a state of total longitudinal momentum P^+ , the choice $L = 2\pi K/P^+$, with K a fixed integer, is sometimes convenient (discrete light-cone quantisation (DLCQ) [2]). Naively, one might try to obtain the corresponding transverse lattice gauge theory by taking the small distance continuum limit of Wilson's lattice gauge theory [4] in the x^+ and x^- directions. However, there are some difficulties associated with this; in particular the light-cone quantisation of the resulting coupled gauged non-linear sigma models at each transverse lattice site is awkward [3]. Bardeen and Pearson [5] suggested that, for large values of the transverse lattice spacing a , a linear sigma model approximation could be used. Instead of formulating the lattice part of the theory in terms of link variables $U \in SU(N)$, one uses link variables $M \in GL(N)$, where the complex matrices M transform in the same way under gauge transformation. Use of such linearized variables is expected to be efficient on coarse lattices, as they represent the important degrees of freedom at such scales. Further discussion of the origin and meaning of these 'colour-dielectric' variables can be traced in ref. [6].

Thus, in transverse lattice gauge theory the Lorentz indices $\mu, \nu \in \{0, 1, 2, 3\}$ are split into LF indices $\alpha, \beta \in \{+, -\}$ and transverse indices $r, s \in \{1, 2\}$. One has link variables $M_r(\mathbf{x})$ associated with the link from \mathbf{x} to $\mathbf{x} + a\hat{\mathbf{r}}$ on a square transverse lattice, together with continuum $SU(N)$ gauge potentials $A_\alpha(\mathbf{x})$ and Dirac fermions $\Psi(\mathbf{x})$ associated with a site \mathbf{x} . These variables transform under transverse lattice gauge transformations $V \in SU(N)$ as

$$A_\alpha(\mathbf{x}) \rightarrow V(\mathbf{x})A_\alpha(\mathbf{x})V^\dagger(\mathbf{x}) + i(\partial_\alpha V(\mathbf{x}))V^\dagger(\mathbf{x})$$

$$M_r(\mathbf{x}) \rightarrow V(\mathbf{x})M_r(\mathbf{x})V^\dagger(\mathbf{x} + a\hat{\mathbf{r}}) \quad (1)$$

$$\Psi(\mathbf{x}) \rightarrow V(\mathbf{x})\Psi(\mathbf{x}) . \quad (2)$$

While a lattice gauge theory at large a may have simple solutions when formulated in terms of disordered variables M , this will cease to be true when a is made small. Although one may only be interesting in solving the theory at large a because of this, traditionally the only way to derive the appropriate effective theory was to start at small a and 'integrate out' short distance degrees of freedom. This problem unfortunately remains largely unsolved in the present case. A more radical approach attempts to remove cut-off artifacts already at large a , by tuning the effective theory to respect the continuum sym-

metries (gauge, Poincaré, chiral) required of the theory. This is the method that will be followed here.

The hamiltonian will consist of all operators invariant under lattice gauge symmetries (2) and Poincaré symmetries unviolated by the cut-offs. We will always have in mind that the transverse lattice spacing a is the only cut-off to which this applies, all others, such as DLCQ, having been extrapolated (in principle). As a result, we may also use dimensional counting in the x^\pm co-ordinates to limit the number of allowed operators. We search for a domain of coupling constants² where Poincaré symmetries violated by the cut-off a are restored. In a general lattice gauge theory, gauge and Poincaré symmetry alone is not sufficient to yield a trajectory of couplings flowing to the continuum QCD. However, the transverse lattice gauge theory is actually ‘half-continuum’ (A^+ and A^- appear explicitly), so it is inconceivable that anything else is obtained provided Lorentz invariance can be convincingly demonstrated. Symmetries that may be ‘spontaneously broken’, such as chiral symmetry, are treated in an unconventional way in light-front quantization. Because cut-offs may, and usually are chosen to, render the vacuum state trivial, symmetries that would conventionally be referred to as spontaneously broken will appear explicitly broken in the light-front hamiltonian. The behaviour of the explicit symmetry-breaking couplings is governed by the underlying symmetry invariance [7] and, in the case of ‘dynamical symmetry breaking’, by stability of the vacuum.

In a real calculation at a given lattice spacing a one must make further approximations, both on the Hamiltonian and its Hilbert (Fock) space. The Fock space at fixed lattice spacing may be truncated with the DLCQ cut-off K and also a Tamm-Dancoff cut-off on the maximum number of quanta in a state. In principle the latter two cut-offs should be extrapolated, as has been done in studies of glueballs [11], if we are to follow the line of argument above. The QCD hamiltonian may be expanded in gauge-invariant powers of M and the quark fields Ψ . The intuitive justification for this is that both should behave like massive degrees of freedom (at large a); the former because of its association with the colour-dielectric mechanism of confinement and the latter because of spontaneous chiral symmetry breaking. We may also assume some transverse locality by expanding the hamiltonian in $a\mathbf{P}$, where \mathbf{P} is transverse momentum. All these approximations have a physical justification and may be systematically relaxed. A final couple of approximations, which may be made for convenience, are to keep as many Poincaré generators as possible in kinematic form (independent of interactions and their renormalisation) and to neglect non-trivial longitudinal momentum dependence of coupling ‘constants’. The consequences of this will show up in the extent to which symmetry can be restored. The above approximations provide a general prescription for reduc-

² We must allow these couplings to be functions of longitudinal momentum p^+ in general [7].

ing the allowed operators in the hamiltonian down to a finite set, which may be systematically enlarged, and reducing the Hilbert space to a finite dimension, which may be extrapolated. The reward for a non-standard quantization with non-standard variables and approximations will be a dramatically simplified hadronic wavefunction.

2 Light meson calculations

Let us consider the first non-trivial approximation to the problem of meson boundstates in transverse lattice QCD. Applying the previous considerations, the leading-order lagrangian for a single quark flavour is

$$\begin{aligned}
L = & \sum_{\mathbf{x}} \int dx^- \sum_{\alpha, \beta=+, -} \sum_{r=1, 2} -\frac{1}{2G^2} \text{Tr}\{F^{\alpha\beta} F_{\alpha\beta}\} \\
& + \text{Tr}\{[(\partial_\alpha + iA_\alpha(\mathbf{x})) M_r(\mathbf{x}) - iM_r(\mathbf{x}) A_\alpha(\mathbf{x} + a\hat{\mathbf{r}})] [\text{h.c.}]\} \\
& - \mu^2 \text{Tr}\{M_r M_r^\dagger\} + i\bar{\Psi} \gamma^\alpha (\partial_\alpha + iA_\alpha) \Psi - \mu_F \bar{\Psi} \Psi \\
& + i\kappa_A \left(\bar{\Psi}(\mathbf{x}) \gamma^r M_r(\mathbf{x}) \Psi(\mathbf{x} + a\hat{\mathbf{r}}) - \bar{\Psi}(\mathbf{x}) \gamma^r M_r^\dagger(\mathbf{x} - a\hat{\mathbf{r}}) \Psi(\mathbf{x} - a\hat{\mathbf{r}}) \right) \\
& + \kappa_S \left(\bar{\Psi}(\mathbf{x}) M_r(\mathbf{x}) \Psi(\mathbf{x} + a\hat{\mathbf{r}}) + \bar{\Psi}(\mathbf{x}) M_r^\dagger(\mathbf{x} - a\hat{\mathbf{r}}) \Psi(\mathbf{x} - a\hat{\mathbf{r}}) \right)
\end{aligned}$$

The Hilbert space of the corresponding light-front hamiltonian will also be expanded in fields. For the present calculation we keep only the $|\bar{\Psi}(\mathbf{x})\Psi(\mathbf{x})\rangle$ and $|\bar{\Psi}(\mathbf{x})M_r(\mathbf{x})\Psi(\mathbf{x}+a\hat{\mathbf{r}})\rangle$ states and their translates. Thus, in this ‘one-link’ approximation, a quark and antiquark can be at the same transverse lattice site, or separated by one link.³ This is the most severe approximation that still allows a meson to propagate on the transverse lattice. The calculations I have done so far have been for fixed K . This cut-off (at least) must eventually be extrapolated.

Using the chiral representation [9] we decompose $\Psi^\dagger = (u_+^*, v_+^*, v_-^*, u_-^*)/2^{1/4}$ into left (right) movers v (u) with a helicity subscript. In light-cone gauge $A_- = 0$, one can eliminate non-dynamical degrees of freedom A_+ and v_\pm at the classical level, to derive a light-front hamiltonian in terms of transverse polarizations only:

$$P^- = \int dx^- \sum_{\mathbf{x}} \frac{G^2}{4} \left(\text{Tr}\left\{J^+ \frac{1}{(i\partial_-)^2} J^+\right\} - \frac{1}{N} \text{Tr}\{J^+\} \frac{1}{(i\partial_-)^2} \text{Tr}\{J^+\} \right)$$

³ This approximation to the Hamiltonian and Fock space was first considered in ref.[8]. Those authors subsequently treated it as a phenomenological model, fixing coupling constants by hand and comparison to experimental mass ratios. However, see also Seal’s talk in these proceedings.

$$+\frac{\mu_F^2}{2}\left(F_+^\dagger\frac{1}{i\partial_-}F_++F_-^\dagger\frac{1}{i\partial_-}F_-\right)+\mu^2\sum_{r=1}^2\text{Tr}\{M_rM_r^\dagger\} \quad (3)$$

$$F_\pm(\mathbf{x})=-u_\pm(\mathbf{x})+\frac{\kappa_S}{\mu_F}\sum_r\left(M_r(\mathbf{x}-a\hat{\mathbf{r}})+M_r^\dagger(\mathbf{x})\right)u_\pm(\mathbf{x}) \\ \pm\frac{i\kappa_A}{\sqrt{2}\mu_F}\left((M_1(\mathbf{x}-a\hat{\mathbf{r}})\mp iM_2(\mathbf{x}-a\hat{\mathbf{r}}))\right. \\ \left.-(M_1^\dagger(\mathbf{x})\pm iM_2^\dagger(\mathbf{x}))\right)u_\mp(\mathbf{x}) \quad (4)$$

$$J^+(\mathbf{x})=i\sum_r\left(M_r(\mathbf{x})\overset{\leftrightarrow}{\partial}_-M_r^\dagger(\mathbf{x})+M_r^\dagger(\mathbf{x}-a\hat{\mathbf{r}})\overset{\leftrightarrow}{\partial}_-M_r(\mathbf{x}-a\hat{\mathbf{r}})\right) \\ +u_+(\mathbf{x})u_+^\dagger(\mathbf{x})+u_-(\mathbf{x})u_-^\dagger(\mathbf{x}) \quad (5)$$

With the exception of DLCQ self-energies, which are logarithmically divergent as $K \rightarrow \infty$ and needed for finite physical answers, the quartic terms in $F^\dagger(i\partial_-)^{-1}F$ are dropped for the following calculation since the one-link approximation treats them asymmetrically [8]. They may be included in higher link approximations and are essential for recovering parity invariance [10]. Let us define $\bar{G} = G\sqrt{(N^2-1)/N}$, which has the dimensions of mass, and introduce the dimensionless variables

$$mb = \frac{\mu}{G} \quad ; \quad mfs1 = \left(\frac{\mu_F^{(1)}}{G}\right)^2 \quad ; \quad mfs2 = \frac{\mu_F^{(2)}}{G} \quad ; \quad (6)$$

$$ka = \frac{\kappa_A}{\bar{G}} \quad ; \quad ks = \frac{\kappa_S}{\bar{G}} \quad . \quad (7)$$

A Fock-sector dependent bare fermion mass $\mu_F^{(i)}$ has been allowed because the one-link approximation produces sector-dependent self-energies. $i = 1$ is the $\bar{\Psi}\Psi$ sector and $i = 2$ corresponds to the $\bar{\Psi}M\Psi$ sector. All N -dependence of the theory has been absorbed into \bar{G} now.

Following a successful series of glueball studies with van de Sande [11,12], I investigated the coupling constant space (6)(7) for signals of enhanced Lorentz covariance using similar methods. The preliminary results I present here are mainly intended as an illustration of the procedure, and in no way should be considered definitive. Ideally, after DLCQ and Tamm-Dancoff cut-offs are extrapolated, the theory should exhibit enhanced Lorentz covariance on an approximately one-dimensional trajectory of couplings, along which a varies. This trajectory represents the best approximation to the (unique) renormalised trajectory in the infinite-dimensional space of couplings. The estimate itself is not unique because it depends on the quantification of violations of Lorentz covariance. Before proceeding one must address chiral symmetry. There are

two related issues. Firstly, cut-offs tend to break chiral symmetry in light-front quantisation because it is a dynamical symmetry. Secondly, spontaneous chiral symmetry breaking cannot appear as a non-trivial vacuum. The first issue means we should allow chiral-symmetry breaking terms in the cut-off hamiltonian. In principle they should be tuned so that the appropriate chiral Ward identities are satisfied. Spontaneous breaking would then have to appear as explicit breaking in the hamiltonian (see [7] for an instructive example). Since the boundstate spectrum is at our disposal, a convenient criterion that addresses both issues at once is 't Hooft's anomaly matching condition [13]. In a Lorentz-covariant theory, chiral symmetry is realised in the boundstate spectrum either as a goldstone boson (massless π) or massless composite fermions. Therefore, we could try to find the renormalised trajectory by optimising both Lorentz covariance and one or other of these chiral symmetry conditions. An incorrect choice for the manifestation of chiral symmetry will make Lorentz covariance difficult to obtain in a theory with a stable vacuum (no tachyons). I used a χ^2 test with variables to measure the anisotropy of π and ρ dispersion relations and the deviation from zero of the pion to ρ mass ratio.⁴ The QCD scale was set from the (experimental) ρ mass and the lattice spacing by demanding isotropy of the π 's dispersion in continuum and on-axis lattice directions. Solving the P^- eigenvalue problem for about 1000 combinations of the couplings (6)(7) and at various momenta, figure 1 shows the result for the one-link approximation at $K = 8$ and a particular link-field mass $mb = 0.2$. The transverse lattice spacing $a = 0.35\text{fm}$ (with a large error). This creates a problem in the one-link approximation, since it is smaller than true π 's radius. The π is artificially squashed in the transverse direction or, in more technical language, the higher Fock state structure will be prone to cut-off artifacts. The $|\bar{\Psi}(\mathbf{x})\Psi(\mathbf{x})\rangle$ sector may be less sensitive to cut-off artifacts, however.

Although the results are still quite crude and further work is needed to search couplings more finely and with more realistic criteria, extrapolate K and add more links/quarks, some results are given here for experimentally known observables. The spin projections of the ρ are not all degenerate because of residual breaking of Lorentz symmetry. Setting the averaged mass to 770 MeV, at the couplings with minimum χ^2 I find $m_\pi = 23$ MeV, $m_\rho(J_z = 0) = 642\text{MeV}$, $m_\rho(J_z = \pm 1) = 898\text{MeV}$.⁵ The $\pi - \rho$ splitting is generated by the helicity-flip term κ_A (see Perry's talk in these proceedings). The $|\bar{\Psi}(\mathbf{x})\Psi(\mathbf{x})\rangle$ component of the meson wavefunction is related to the leading order perturbative QCD expression for many exclusive processes [14]. Fitting the transverse lattice pion wavefunction to the conformal expansion [15,16] of this component, one finds

⁴ To complete the argument one should also study baryons. This requires more links and fermions for lattice propagation and chiral anomalies.

⁵ As a consistency check on the scale, the string tension calculated in the same approximation is found to be $\sqrt{\sigma} = 0.643m_\rho = 495\text{MeV}$.

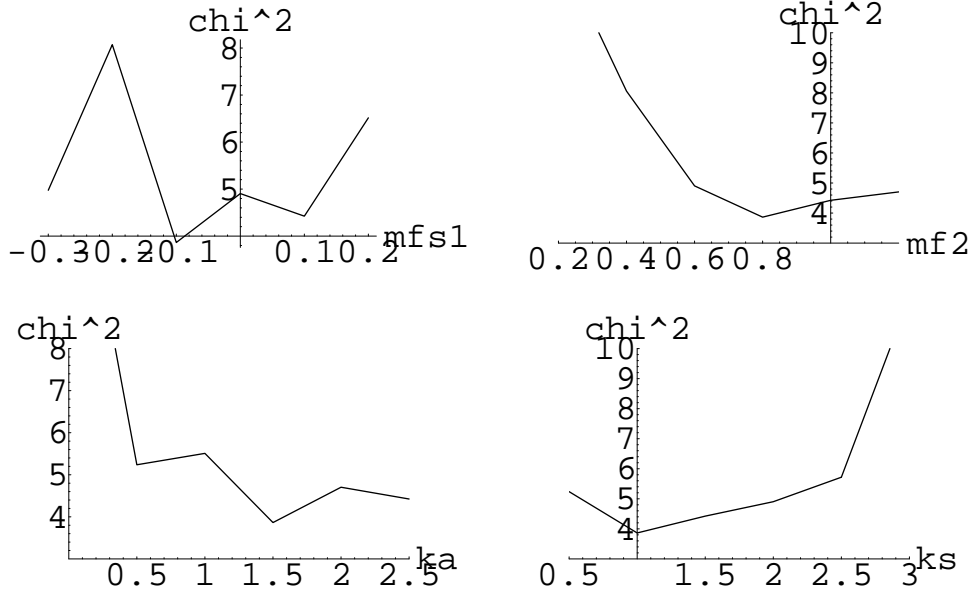


Fig. 1. χ^2 test of Lorentz and chiral symmetry as a function of couplings for $mb = 0.2$.

$$\phi_\pi(x, Q^2 \sim 1\text{GeV}^2) = \frac{2.653}{a} x(1-x) \left\{ C_0^{3/2} + 0.237 C_2^{3/2} - 0.102 C_4^{3/2} - 0.05 C_6^{3/2} + \dots \right\}. \quad (8)$$

The transverse scale 1GeV is a rough estimate based on π/a , and $C_n^{3/2}(1-2x^2)$ are the appropriate Gegenbauer polynomials. Eq. (8) directly confirms that the conformal expansion is a good one. The overall normalisation yields $f_\pi = 101\text{MeV}$ compared with the experimental value $f_\pi(\text{exp.}) = 93\text{MeV}$. Using leading order perturbative QCD evolution to other scales Q^2 , and assuming leading order perturbative QCD factorization, we can compare (8) with direct and indirect experimental measurements. The cleanest extraction from a π form factor comes from $\gamma\gamma^* \rightarrow \pi^0$; $Q^2 F_{\pi^0}$ is approximately constant in the range $1\text{GeV}^2 < Q^2 < 10\text{GeV}^2$ and at the higher end $Q^2 F_{\pi^0}(Q^2 = 8\text{GeV}^2) = 0.16 \pm 0.03\text{GeV}$ has been measured at CLEO-II [17]. This compares with the

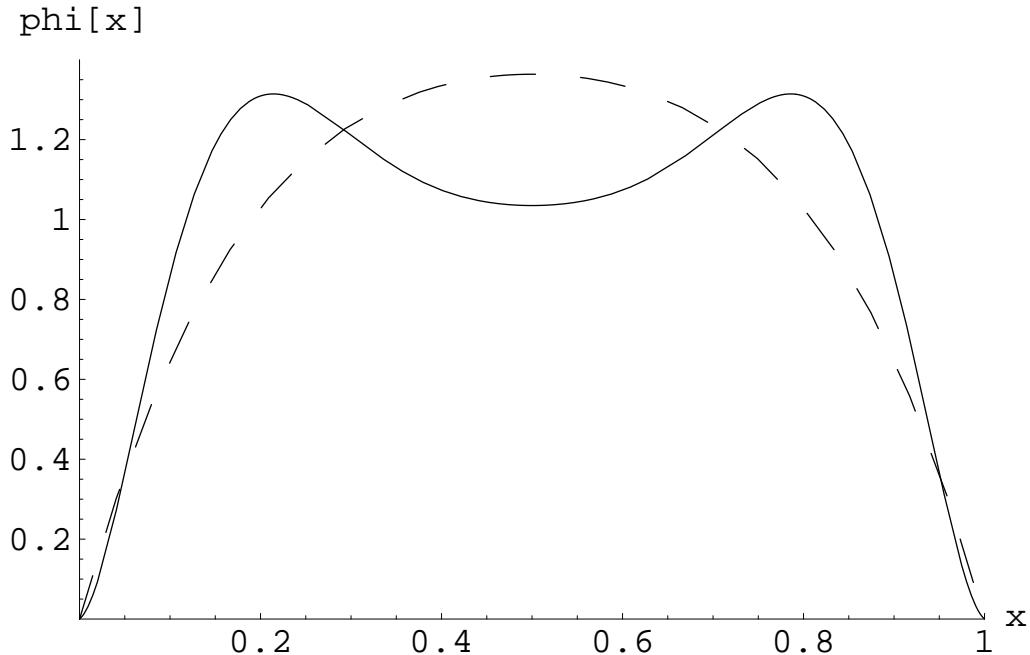


Fig. 2. The pion valence quark amplitude at zero transverse separation and $Q^2 \sim 10 \text{ GeV}^2$. Solid line is theory, chain line is experimental deduction (both with unknown error!). Both are normalised to area 1 for this comparison.

theoretical result

$$Q^2 F_{\pi^0}(Q^2 = 8 \text{ GeV}^2) = \frac{4}{\sqrt{3}} \int_0^1 dx \frac{\phi_\pi(x, Q^2 \sim 8 \text{ GeV}^2)}{x} = 0.21 \text{ GeV} \quad (9)$$

Direct tests of ϕ_π have recently become possible from diffractive dissociation on a nucleus $\pi + A \rightarrow A + \text{jets}$ [18]. Figure 2 compares the quark amplitude with the one quoted in ref.[18] as that which best fits the jet data after hadronization and experimental acceptance (see Ashery's talk in these proceedings for an updated account of this experiment.) Although hadronization tends to wash out any fine structure in the quark amplitude, making true comparison between similar curves difficult, the theoretical curve seems a bit too peaked away from $x = 1/2$. This is consistent with the result (9) being

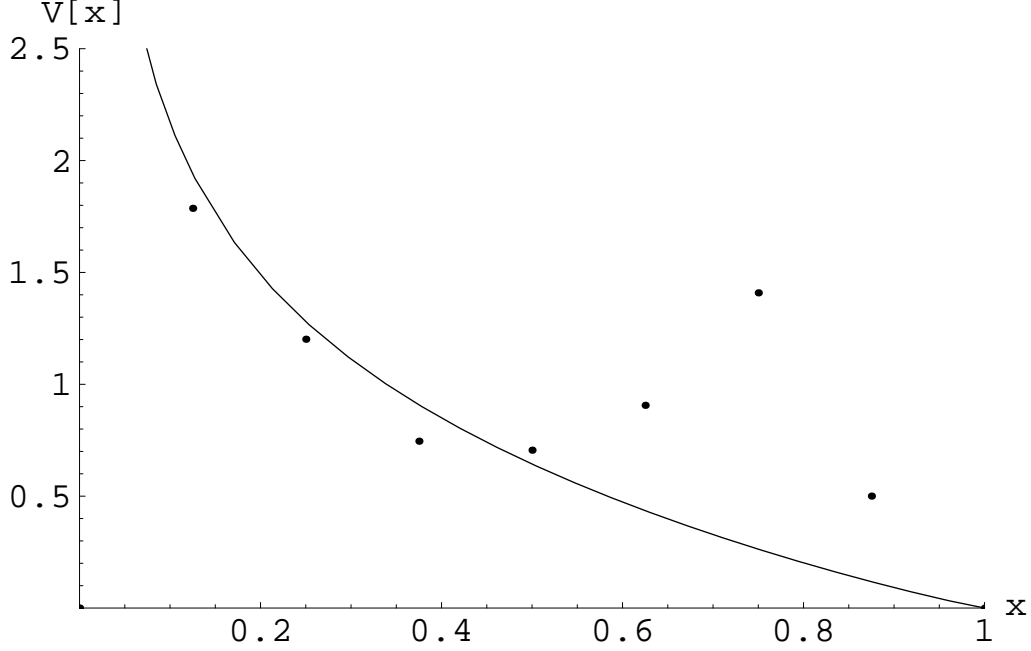


Fig. 3. The non-singlet quark probability distribution V of the pion at $Q^2 \sim 4\text{GeV}^2$. Solid line is a model fit to experiment, data points are from the transverse lattice wavefunction with leading order evolution.

slightly too high. Of course there is no reason why the finite- K calculation should give exactly the right result. Use of a nearly massless pion, ambiguous normalisation scale Q^2 , leading order evolution from 1 to 10GeV^2 , etc. leads to further errors.

Inclusive processes are typically sensitive to higher Fock state structure. Here, we may not expect to do so well because of the severity of our cut-offs. This suspicion is confirmed by the standard $x^\alpha(1-x)^\beta$ fits to the non-singlet quark probability distribution in the π , obtained from πN Drell-Yan [19]; see fig.3. The deviation of the lattice results at large x is due at least partly to cut-off artifacts. Relaxing the DLCQ and one-link approximation will allow some of the quark momentum at large x to radiate into small x quarks and gluons as more gluonic channels are opened. The gluons carry only about 10% of the π momentum at present — much lower than the accepted experimental value

[20] — confirming there is a problem with higher Fock states.

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